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Numerical renormalization-group approach to a sandpile

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Multiple topplings and grain redistribution are two essential features in sandpile dynamics. A renormalization group (RG) approach incorporating these features is investigated. The full enumeration of all relaxations involving such an approach is difficult. Instead, we developed an efficient procedure to sample the relaxations. We applied this RG scheme to a square lattice and a triangular lattice. As shown by the fixed point analysis on a square lattice, the effect of multiple topplings leads the resultant height probabilities towards the exact solution while the effect of grain redistribution does not.

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I. INTRODUCTION

Self-organized criticality (SOC) [1] provides a possible pathway for exploiting the underlying mechanism of scaling behavior in many natural phenomena. From a theoretical aspect, only a few rigorous solutions of SOC models were found [2,3]. On the other hand, the renormalization group (RG) approach, which is based on loss of scale at a fixed point [4], is a theoretical tool for investigating a system with scaling behavior. It has been widely applied to critical phenomena. Therefore, employing the RG approach in analyzing SOC may have the potential to benefit the theoretical work on SOC.

The Bak-Tang-Wiesenfeld (BTW) sandpile [5] is a prototype of SOC. In 1994, Pietronero, Vespignani, and Zapperi (PVZ) [6] proposed the so-called dynamically driven RG scheme for the BTW sandpile. It was shown on a square lattice where critical exponents are faithfully estimated by counting the relaxations for the map between the 2×2 RG cells of different scales. Subsequently, Ivashkevich [7] has furnished considerable refinements for the PVZ's RG scheme. From his study, not only the exponents but also the height probabilities can be obtained. More recently, the dynamically driven RG has been applied to a triangular lattice [8] and the directed sandpile [9]. In addition, the RG equation based on cross-shaped RG cells [10] was also studied.

In the above studies, RG equations were established using the similarity of topplings observed at different scales. However, the essential feature of the BTW sandpile, multiple topplings, and grain redistribution [11] were seldom considered. In the present study we attempted a probable refinement of the RG scheme when multiple topplings and grain redistribution are considered. Minimal design modifications were made to the RG framework provided by PVZ and Ivashkevich such that the RG scheme became permissible. In practice, when multiple topplings and grain redistribution are considered, the number of relaxations will increase and become eventually unbounded. This renders the counting method employed earlier useless. Therefore, in place of the earlier counting methods [6,7], we developed a technique of random sampling which proved to be efficient as well as reliable. Subsequently, considerable improvement was obtained for the height probabilities on a square lattice. Finally, in order to exhibit the potential of this RG method, we also applied this RG scheme to the sandpile on a triangular lattice.

II. RG PARAMETERS

In order to express an RG scheme, it is necessary to develop conventions to implement the changing of scale. Consider a square lattice. Each square domain with area size $\Delta^{(\lambda)}=2^{\lambda}\times 2^{\lambda}$ is called a λ square and is positioned at i. Accordingly, the kth location of the nearest neighbor (NN) of λ square i can be conveniently labeled by i_k and shown in Fig. 1(a) for $k=1$, 2, 3, and 4. The dynamics based on the RG essentially follow the instruction of the original BTW sand-

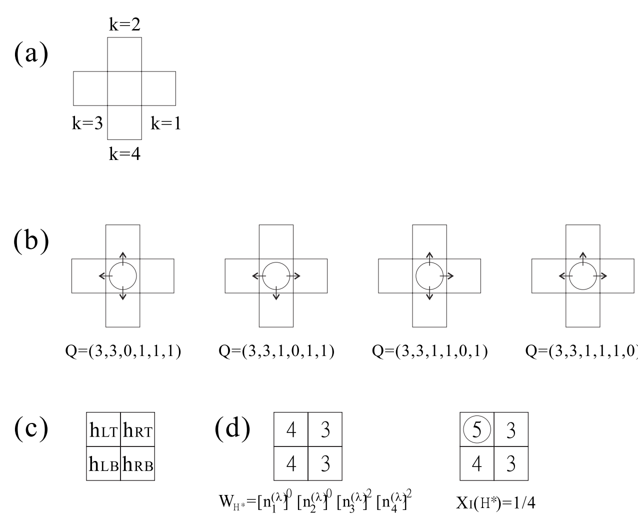

FIG. 1. (a) Construction of a λ square i and its NNs i_k for $k=1$, 2, 3, and 4. (b) All Q corresponding to T_3. A λ square labeled by a circle will transfer λ grains to its NNs according to the arrows. (c) A cell containing four λ squares which are labeled by LT, LB, RT, and RB. The height of a λ square is printed inside the square. (d) W_{H^*} for $h_{LT}=4$, $h_{LB}=4$, $h_{RT}=3$, and $h_{RB}=3$. (e) $O_I H^*$ and $X_I(H^*)$ for $I=$LT, where H^* is obtained from (d).

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pile which corresponds to the zero-scale prescription of the RG scheme. For the zero scale, each location i is equipped with a height h_i which represents the grain number at zero square i. If $h_i > 4$ for one specific i, then the zero square i will topple through

$$h_i \to h_i - 4,$$
$$h_{i_k} \to h_{i_k} + 1. \tag{1}$$

Initially, $\{h_i\}$ is stable, i.e., h_i=1, 2, 3 or 4. The sandpile dynamics are triggered by adding one grain to a randomly chosen zero square and its height increases by one. If the height of the chosen zero square exceeds 4, then the system evolves by topplings until all $h_i \leq 4$ and the toppled zero squares form an avalanche with an area $A^{(0)}$, which is the distinct zero squares that toppled. It is generally believed that the probability distribution of the avalanche area $\mathcal{P}(A^{(0)})$ follows $\mathcal{P}(A^{(0)}) \sim (A^{(0)})^{-\tau}$, where τ is the area exponent.

For $\lambda \geq 1$, an effective representative of the height set associated with each location i must be defined such that the rule of toppling can be furnished at this scale. Let us imagine that the topplings are similar at each scale. In the λ square i, the height h_i is represented by counting the effective grains, say λ grains, aggregated in this area. Basically, λ square i with $1 \leq h_i \leq 4$ is referred to as stable as long as all the underlying zero squares remain stable. Otherwise, if $h_i > 4$, it indicates that some underlying zero squares are not stable. This will induce a series of topplings of zero squares taking place within the domain of λ square. Note that the construction of λ grains is essentially in response to the topplings of zero squares. Suppose that d_k zero grains contributed by λ square i are sent to λ square i_k. We approximate such a process as λ square i toppling by losing $\hat{q}$ λ grains and rendering q_k λ grains for each NN λ square i_k, namely,

$$h_i \to h_i - \hat{q},$$
$$h_{i_k} \to h_{i_k} + q_k, \tag{2}$$

where we set $\hat{q} \geq 1$, q_k=1 as $d_k > 0$, and q_k=0 as d_k=0. When

$$r = \sum_{k=1}^{4} q_k, \tag{3}$$

we end up with an r-directional toppling denoted by T_r. Therefore, a toppling can explicitly be listed in terms of the sextuple

$$Q = (r, \hat{q}, q_1, q_2, q_3, q_4). \tag{4}$$

For a fixed (q_1, q_2, q_3, q_4), the value of $\hat{q}$ in Eqs. (2) and (4) has not been determined yet. In this paper, we assign

$$\hat{q} = \begin{cases} r & \text{for } r \geq 1, \\ 1 & \text{for } r = 0. \end{cases} \tag{5}$$

For $r \geq 1$, we follow the setting of Ref. [6] where $\hat{q} - \Sigma q_k = 0$ corresponds to λ grain conservation. The case r=0 corresponds to $Q = (0, \hat{q}, 0, 0, 0, 0)$, which refers to the break of the λ grain conservation, i.e., $\hat{q} - \Sigma q_k = \hat{q} \neq 0$. The sandpile system on the λth stage will then dissipate $\hat{q}$ λ grains through a toppling of T_0. From a physical perspective, r=0 should essentially minimize the $\hat{q}$ value. Since $\hat{q} \geq 1$ must hold, the convenient choice will be $\hat{q}$=1 as r=0. This setting is consistent with Ref. [11] and is checked numerically by the second test in Sec. IV. Based on the construction of NNs for a λ square in Fig. 1(a), we list all Q with r=3 in Fig. 1(b). In practice, m_r sorts of Q, say m_0=1, m_1=4, m_2=6, m_3=4, and m_4=1, are found for each r.

Two more aspects appear to be crucial for the subsequent analyses. Before triggering a sandpile evolution of the λth stage, every height is stable and the probability of the height of one λ square being j is $n_j^{(\lambda)}$. Then, the property of height for one λ square is described as

$$\vec{n}^{(\lambda)} = (n_1^{(\lambda)}, n_2^{(\lambda)}, n_3^{(\lambda)}, n_4^{(\lambda)}) \text{ with } \sum_{j=1}^{4} n_j^{(\lambda)} = 1. \tag{6}$$

During a sandpile evolution of the λth stage, the probability of a toppling being T_r is $p_r^{(\lambda)}$. The property of topplings for one λ square is described as

$$\vec{p}^{(\lambda)} = (p_0^{(\lambda)}, p_1^{(\lambda)}, p_2^{(\lambda)}, p_3^{(\lambda)}, p_4^{(\lambda)}) \text{ with } \sum_{r=0}^{4} p_r^{(\lambda)} = 1. \tag{7}$$

One T_r corresponds to m_r kinds of Q. We assume that the probability of occurrence of a specified Q is $(p_r^{(\lambda)}/m_r)$.

Extending the idea of Ivashkevich [7], one can obtain the relationship between $\vec{n}^{(\lambda)}$ and $\vec{p}^{(\lambda)}$. Consider a λ square i with $h_i = \alpha$ for $1 \leq \alpha \leq 4$. After adding one λ grain to this λ square, it finally evolves to $h_i = \beta$ as follows:

$$\beta = \begin{cases} \alpha + 1 & \text{for } \alpha \leq 3, \\ \alpha + 1 - \hat{q} = \begin{cases} 4 & \text{for } \alpha = 4 \text{ and } r = 0, \\ 5 - r & \text{for } \alpha = 4 \text{ and } r \geq 1, \end{cases} \end{cases} \tag{8}$$

where there is no toppling for $\alpha \leq 3$ and a toppling through T_r happens for α=4. For example, there are two ways to end up with β=3; by means of (i) α=2 with probability $n_2^{(\lambda)}$ and (ii) α=4 and r=2 with probability $n_4^{(\lambda)} p_2^{(\lambda)}$. Items (i) and (ii) increase the density of h_i=3. On the contrary, item (iii) α=3 with probability $n_3^{(\lambda)}$ brings the height away from h_i=3. Then item (iii) decreases $n_3^{(\lambda)}$. Based on the concept of master equation, the rate of change of $n_3^{(\lambda)}$, i.e., $dn_3^{(\lambda)}/dt$, essentially consists of items (i), (ii), and (iii), i.e., $dn_3^{(\lambda)}/dt \sim n_2^{(\lambda)} + n_4^{(\lambda)} p_2^{(\lambda)} - n_3^{(\lambda)}$.

In general, we can write

$$dn_j^{(\lambda)}/dt \sim n_{j-1}^{(\lambda)} + n_4^{(\lambda)} \theta_{5-j} - n_j^{(\lambda)}, \tag{9}$$

where $n_0^{(\lambda)}$=0, $\theta_1 = p_0^{(\lambda)} + p_1^{(\lambda)}$, and $\theta_j = p_j^{(\lambda)}$ for $j \geq 2$. Put the steady-state assumption $dn_j^{(\lambda)}/dt$=0 into Eq. (9). Then the relationship between $\vec{n}^{(\lambda)}$ and $\vec{p}^{(\lambda)}$ is given by

$$n_j^{(\lambda)} = \sum_{j^+=4,-1}^{5-j} \theta_{j^+} \Bigg/ \sum_{j^+=1}^{4} j^+ \theta_{j^+}. \tag{10}$$

A λ square will lose $\overline{p^{(\lambda)}} = (p_0^{(\lambda)} + p_1^{(\lambda)} + 2p_2^{(\lambda)} + 3p_3^{(\lambda)} + 4p_4^{(\lambda)})$ λ grains on average per toppling. Then, from the balance be-

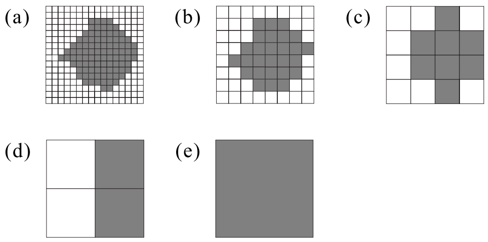

FIG. 2. An avalanche represented by λ squares. The λ square filled in gray is a toppled λ square. (a) $A^{(0)}=96$, (b) $A^{(1)}=27$, (c) $A^{(2)}=8$, (d) $A^{(3)}=2$, (e) $A^{(4)}=1$.

tween the input flow and output flow of a λ square, we have $n_4^{(\lambda)}=1/\overline{p^{(\lambda)}}$, which is exactly the same as Eq. (10) for $j=4$. Note that Eq. (10) for $j=4$ is identical to the corresponding equation [12] of Ref. [11] where T_0 is considered. If T_0 is not considered, we must set $p_0^{(\lambda)}=0$ and then Eq. (10) is identical to the corresponding equation in Ref. [7].

III. RG DYNAMICS AND MAPPING RULE

In view of RG, the correspondence of toppling observed at different scales plays a fundamental role. According to the mapping rule [6], a $(\lambda+1)$ square is referred to as a toppled $(\lambda+1)$ square if the toppled λ squares inside a $(\lambda+1)$ square span this $(\lambda+1)$ square from top to bottom or right to left. Otherwise, this $(\lambda+1)$ square is a nontoppled $(\lambda+1)$ square. Examples are shown in Figs. 2(a)–2(e) for $\lambda=0$, 1, 2, 3, and 4, respectively, where $A^{(\lambda)}$ is the number of toppled λ squares. If the $A^{(0)}$ in Fig. 2(a) decreases, the toppled λ square for $\lambda=4$, as shown in Fig. 2(e), may become a nontoppled λ square. Hence, by substantially reducing $A^{(0)}$, the chance of forming a toppled λ square is diminished. Moreover, it is well recognized that the shape of a toppling area on $\lambda=0$ stage is always compact and disklike [6], e.g., Fig. 2(a). Thus, a threshold, above which the avalanche area $A^{(0)}$ can induce the toppled λ squares, is expected to be roughly proportional to domain size $\Delta^{(\lambda)}$ [6], namely, $A^{(\lambda)}\geqslant 1$ for $A^{(0)}\geqslant b\Delta^{(\lambda)}$ and $A^{(\lambda)}=0$ for $A^{(0)}<b\Delta^{(\lambda)}$, where b refers to a certain constant less than unity, as illustrated in Fig. 2(e).

The RG dynamics are essentially comprised of relaxation procedures at each definite scale, which emerge as evolutions of height configurations of sets of λ squares. Take for instance a cell consisting of 2×2 λ squares located at the left top (LT), left bottom (LB), right top (RT), and right bottom (RB), as shown in Fig. 1(c). A height configuration is prescribed by the set $H=(h_{\mathrm{LT}},h_{\mathrm{LB}},h_{\mathrm{RT}},h_{\mathrm{RB}})$. The initial height configuration $H=H^*$, e.g., in Fig. 1(d), is eventually determined as stable. If a λ square $I\in\{\mathrm{LT},\mathrm{LB},\mathrm{RT},\mathrm{RB}\}$ called the initial λ square receives one λ grain from outside, then the height configuration changes from $H=H^*$ to $H=O_IH^*$, where O_I is an operator denoting $h_I\rightarrow h_I+1$. Consider the case O_IH^* is not stable. Then a relaxation procedure follows, as shown in Figs. 3(a)–3(c),

$$O_IH^* = H_1 \xrightarrow{[I_1,Q_1]} H_2 \xrightarrow{[I_2,Q_2]} \cdots \xrightarrow{[I_N,Q_N]} H_{N+1} = H^{**}, \tag{11}$$

where N counts the steps of topplings within a relaxation procedure and $H=H^{**}$ is stable and called the final height

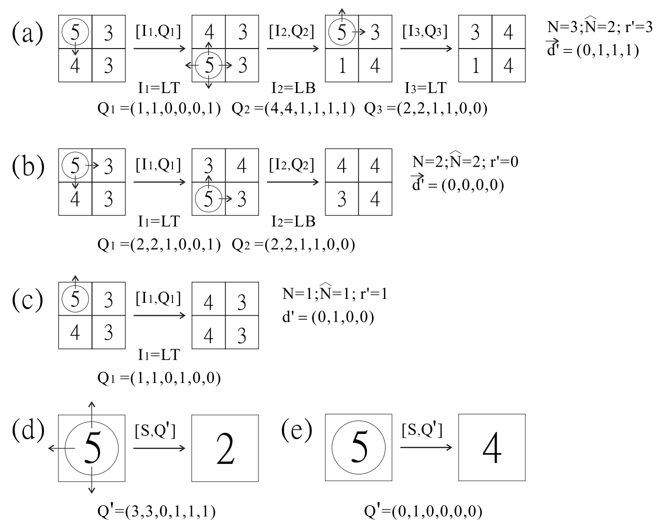

FIG. 3. Relaxations described by Eqs. (11) and (13) where H^* and I are obtained from Figs. 1(d) and 1(e). (a) Multiple topplings. The toppled λ square LT topples twice where $Y_\Phi(H^*,I)=[p_1^{(\lambda)}/4]\times[p_2^{(\lambda)}/6][p_4^{(\lambda)}/1]$. (b) Grain redistribution. The unstable cell evolves to a stable cell through the redistribution of λ grains where $Y_\Phi(H^*,I)=[p_2^{(\lambda)}/6]^2$. (c) The toppled λ squares do not span a 2×2 cell where $Y_\Phi(H^*,I)=[p_1^{(\lambda)}/4]$. (d) The relaxation shown in (a) described by Eq. (13). (e) The relaxation shown in (b) described by Eq. (13).

configuration. The system then evolves by means of a sequence of topplings individually described by $[I_l,Q_l]$ for $1\leqslant l\leqslant N$, where $I_l\in\{\mathrm{LT},\mathrm{LB},\mathrm{RT},\mathrm{RB}\}$ denotes the toppling position within the cell and $Q_l=(r_l,,\hat{q}_l,q_{1l},q_{2l},q_{3l},q_{4l})$ denotes one specified Q.

It is worth noting that contrary to the previous treatments [6,7], the successive toppling steps need not be self-avoiding during a relaxation. In practice, the value of N can be unbounded and $N\geqslant\hat{N}$, where $\hat{N}$ is the number of distinct toppled λ squares. When $N>\hat{N}$, it is recognized that some of the λ squares topple at least twice. This case is appropriate for the so-called multiple topplings, as exemplified by Fig. 3(a).

The activation of the $(\lambda+1)$ squares essentially follows the mapping rule mentioned above. For instance, the toppled λ squares shown in Fig. 3(a) span the 2×2 cell and then this cell is represented by a toppled $(\lambda+1)$ square. Each corresponding relaxation must be equipped with a toppling path

$$\Phi = \{[I_l,Q_l]\} \tag{12}$$

and $\vec{d}'=(d_1',d_2',d_3',d_4')$, where d_k' λ grains are sent to the kth location of NNs of the cell constructed by λ squares LT, LB, RT, and RB. A relaxation on the λth stage guiding a cell to a toppled $(\lambda+1)$ square is called an active relaxation and its associated Φ is called an active Φ denoted by Φ_a. Thus, the Φ of Figs. 3(a) and 3(b) are Φ_a. On the other hand, the Φ of Fig. 3(c) is not a Φ_a and this cell is not considered as a toppled $(\lambda+1)$ square.

For $\Phi=\Phi_a$, we approximate Eq. (11) as a $(\lambda+1)$ square labeled by S topples through $Q=Q'=(r',\hat{q}',q_1',q_2',q_3',q_4')$ expressed as

$$O_S h_S^* \xrightarrow{[S,Q']} h_S^{**}, \tag{13}$$

where h_S^* and h_S^{**} are the initial and final heights of $(\lambda+1)$ square S, respectively, and O_S is an operator for $O_S h_S^*=h_S^*+1$. Notably, the requirements of $h_S^*=4$, $O_S h_S^*=5$, $h_S^{**}\leqslant 4$, and $\hat{q}'=O_S h_S^*-h_S^{**}\geqslant 1$ must be fulfilled. The determination of Q' according to $q_k'=1$ for $d_k'>0$ and $q_k'=0$ for $d_k'=0$ is similar to the discussion between q_k and d_k in Eq. (2). Thus, Eq. (13) represents an r'-directional toppling of $(\lambda+1)$ square S with $r'=\Sigma_k q_k'$.

Based on the above, relaxations described by Figs. 3(a) and 3(b) correspond to the topplings described by Figs. 3(d) and 3(e), respectively. The example provided by Fig. 3(b) refers to the fact that the total λ grain number of $O_I H^*$ equals that of H^{**}. However, $O_I H^*\neq H^{**}$. In other words, the λ grains redistribute inside the RG cell and no λ grain is sent outside of the RG cell. This is called grain redistribution and corresponds to Eq. (13) with $Q'=(0,1,0,0,0,0)$ from Eq. (5).

Technically, each relaxation characterized by Eq. (11) is well marked by a record (H^*,I,Φ). The probability for any definite relaxation is constructed in terms of $\vec{n}^{(\lambda)}$ and $\vec{p}^{(\lambda)}$. First, the probability of the occurrence of an initial height configuration within a cell, as listed in Figs. 1(c) and 1(d), is weighted by $W_{H^*}=[n_{h_{\mathrm{LT}}}^{(\lambda)}][n_{h_{\mathrm{LB}}}^{(\lambda)}][n_{h_{\mathrm{RT}}}^{(\lambda)}][n_{h_{\mathrm{RB}}}^{(\lambda)}]$. Second, the probability of picking a specific I is $X_I(H^*)=1/4$ because there are only four choices, i.e., $I=$LT, LB, RT, or RB as shown in Fig. 1(e). Furthermore, for a given initial height configuration H^* and a given I, the probability of evolving through a specific toppling path $\Phi=\{[I_l,Q_l]\}$ is $Y_\Phi(H^*,I)=\Pi_{l=0}^N[p_{r_l}^{(\lambda)}/m_{r_l}]$, where $Q_l=(r_l,\hat{q}_l,q_{1l},q_{2l},q_{3l},q_{4l})$ corresponds to probability $p_{r_l}^{(\lambda)}/m_{r_l}$. Here, we assumes that $Y_\Phi(H^*,I)$ is independent of I_l. Thus, a relaxation described by Eq. (11) has a probability of

$$C_{H^*,I,\Phi}=W_{H^*}X_I(H^*)Y_\Phi(H^*,I), \tag{14}$$

where $\Sigma_{H^*}\Sigma_I\Sigma_\Phi C_{H^*,I,\Phi}=1$.

IV. RG EQUATIONS AND A SIMPLE SAMPLING TECHNIQUE

The RG transformation between Eqs. (11) and (13) is launched from the issue of probability. Note that only active relaxations will take part in the RG transformation. Samplings of Eq. (11) are useful only for the records implementing active relaxations. Thus, the probabilities of interest should be renormalized accordingly. The (H^*,I,Φ) of an active relaxation is denoted by (H_a^*,I_a,Φ_a). We group all different H_a^* into a set $\{H_a^*\}$. In addition, all different I_a generated from one fixed H_a^* are stored in a set $\{I_a(H_a^*)\}$. Moreover, all Φ_a generated by a fixed H_a^* and a fixed I_a are collected into a set $\{\Phi_a(H_a^*,I_a)\}$. This implies that $\Sigma_{H^*\in\{H_a^*\}}\Sigma_{I\in\{I_a(H_a^*)\}}\times\Sigma_{\Phi\in\{\Phi_a(H_a^*,I_a)\}}C_{H^*,I,\Phi}<1$. By reweighing W_{H^*}, X_I, Y_Φ, and $C_{H^*,I,\Phi}$ to

$$W_{H^*}^{\#}=W_{H^*}\bigg/\left[\sum_{H^*\in\{H_a^*\}}W_{H^*}\right],$$

$$X_I^{\#}(H^*)=X_I(H^*)\bigg/\left[\sum_{I\in\{I_a(H_a^*)\}}X_I(H^*)\right],$$

$$Y_\Phi^{\#}(H^*,I)=Y_\Phi(H^*,I)\bigg/\left[\sum_{\Phi\in\{\Phi_a(H_a^*,I_a)\}}Y_\Phi(H^*,I)\right],$$

$$C_{H^*,I,\Phi}^{\#}=W_{H^*}^{\#}X_I^{\#}(H^*)Y_\Phi^{\#}(H^*,I), \tag{15}$$

we achieve the normalized condition $\Sigma_{H^*\in\{H_a^*\}}\Sigma_{I\in\{I_a(H_a^*)\}}\times\Sigma_{\Phi\in\{\Phi_a(H_a^*,I_a)\}}C_{H^*,I,\Phi}^{\#}=1$.

Each active relaxation on the λth stage is equipped with $r'=\Sigma_{k=1}^4 q_k'$, which will be referred to a $T_{r'}$ on the $(\lambda+1)$th stage. We divide $\{\Phi_a(H_a^*,I_a)\}$ into the subsets $\{\Phi_a^{(r')}(H_a^*,I_a)\}$ subject to definite toppling directions where $\{\Phi_a(H_a^*,I_a)\}=\cup_{r'=0}^4\{\Phi_a^{(r')}(H_a^*,I_a)\}$. In brief, the normalized probability of the r'-directional toppling $p_{r'}^{(\lambda+1)}$ is expressed in terms of

$$p_{r'}^{(\lambda+1)}=\sum_{H^*\in\{H_a^*\}}W_{H^*}^{\#}\left[\sum_{I\in\{I_a(H_a^*)\}}X_I^{\#}(H^*)\left(\sum_{\Phi\in\{\Phi_a^{(r')}(H_a^*,I_a)\}}Y_\Phi^{\#}\right)\right], \tag{16}$$

where $\Sigma_{r'=0}^4 p_{r'}^{(\lambda+1)}=1$. Note that the right-hand side of Eq. (16) is a function of $\vec{p}^{(\lambda)}$ since $W_{H^*}^{\#}$ will be a function of $\vec{p}^{(\lambda)}$ through the transformation in Eq. (10), $X_I^{\#}(H^*)$ is a number, and $Y_\Phi^{\#}$ is also a function of $\vec{p}^{(\lambda)}$. Equation (16) describes the relationship between $\vec{p}^{(\lambda+1)}$ and $\vec{p}^{(\lambda)}$.

There is no barrier for preparing $\{H_a^*\}$ on a 2×2 cell because the total number of H^* is only 4^4 and the number of the candidates for $H^*\in\{H_a^*\}$ is much less than 4^4. Thus, the analytic forms of $W_{H^*}^{\#}$ and $X_I^{\#}(H^*)$ of Eq. (15) are accessible. However, contrary to the directly achievable $Y_\Phi^{\#}(H^*,I)$ in the previous treatment [7,13], obstructions occur in the corresponding enumeration due to multiple topplings and grain redistribution. This is because $Y_\Phi^{\#}(H^*,I)$ requires all information of $\{\Phi_a(H_a^*,I_a)\}$ which may contain an infinite number of elements. Therefore, we consider in turns the total normalized probability of active relaxations generated from a fixed $H^*\in\{H_a^*\}$ and a fixed $I\in\{I_a(H_a^*)\}$, corresponding to the r'-directional toppling of $(\lambda+1)$ square S

$$Z_{r'}^{\#}(H^*,I)=\sum_{\Phi\in\{\Phi_a^{(r')}(H_a^*,I_a)\}}Y_\Phi^{\#}=\frac{\sum_{\Phi\in\{\Phi_a^{(r')}(H_a^*,I_a)\}}Y_\Phi}{\sum_{\Phi\in\{\Phi_a(H_a^*,I_a)\}}Y_\Phi}, \tag{17}$$

where $\Sigma_{r'=0}^4 Z_{r'}^{\#}(H^*,I)=1$. If $\vec{p}^{(\lambda)}$ is known, $Z_{r'}^{\#}(H^*,I)$ can be estimated by the simple sampling technique described in the next paragraph.

In order to determine the value of $\vec{p}^{(\lambda+1)}$ through Eq. (17) for a known $\vec{p}^{(\lambda)}$, the procedure is as follows: (i) Prepare an H^* and an I. Set $(z_0,z_1,z_2,z_3,z_4)=(0,0,0,0,0)$, $\eta(H^*,I)=0$, and $t=0$, where t is a counter for the number of triggers. (ii)

The value of t increases by one, i.e., $t \rightarrow t+1$. For the prepared (H^*,I), start the sandpile dynamics by using the toppling rule listed in Eq. (2). When a λ square topples, a specified Q_l of Eq. (11) is randomly assigned to its toppling by probability $p_{r_l}^{(\lambda)}/m_{r_l}$. The dynamics will stop when all λ squares are stable simultaneously. If the generated relaxation is an active relaxation, record $Q'=(r',\hat{q}',q_1',q_2',q_3',q_4')$ of Eq. (13) after mapping Eq. (11) to Eq. (13), set $\eta(H^*,I)=1$, and force $z_{r'} \rightarrow z_{r'}+1$. (iii) Repeat step (ii) until $t=t_c$ where t_c is the threshold of t. The index $\eta(H^*,I)$ is used to determine whether there is any active relaxation among the relaxations generated. After the simple sampling procedure is finished, $\eta(H^*,I)=1$ denotes that $H^* \in \{H_a^*\}$ and $I \in \{I_a(H_a^*)\}$. Otherwise, $\eta(H^*,I)=0$ denotes $I \notin \{I_a(H_a^*)\}$. For $\eta(H^*,I)=1$, since all nonactive relaxations are not counted in (ii), $z_{r'}/(\Sigma_{r'=0}^4 z_{r'})$ is a normalized probability and we expect $Z_{r'}^{\#}(H^*,I) \approx z_{r'}/(\Sigma_{r'=0}^4 z_{r'})$ if t_c is large enough.

The simple sampling will also benefit the calculations of $W_{H^*}^{\#}$ and $X_I^{\#}(H^*)$ from a numerical aspect. An index $\omega(H^*)=1$ or 0 for $\Sigma_{I=\mathrm{LT,LB,RT,RB}}\,\eta(H^*,I)>\text{or}=0$ is provided to judge if $H^* \in \{H_a^*\}$ or $H^* \notin \{H_a^*\}$, respectively. Thus,

$$W_{H^*}^{\#} = \omega(H^*)W_{H^*} \bigg/ \left[\sum_{H^*} \omega(H^*)W_{H^*}\right], \qquad (18)$$

and

$$X_I^{\#}(H^*) = \begin{cases} \eta(H^*,I) \bigg/ \left[\displaystyle\sum_{I=\mathrm{LT,LB,RT,RB}} \eta(H^*,I)\right] & \text{for } \omega(H^*)=1 \\ 0 & \text{for } \omega(H^*)=0. \end{cases} \qquad (19)$$

For a given $\vec{p}^{(\lambda)}$, the value of $\vec{p}^{(\lambda+1)}$ will be achieved by treating Eqs. (17)–(19) for the simple sampling procedure.

In this paper, we start all RG calculations on a square lattice by setting $\vec{p}^{(0)}=(0,0,0,0,1)$ and $t_c=10\,000$. By estimating $W_{H^*}^{\#}$, $X_I^{\#}(H^*)$, and $Z_{r'}^{\#}(H^*,I)$ through the simple sampling procedure, we obtain the value of $\vec{p}^{(1)}$. Meanwhile, $\vec{p}^{(2)}$ is achievable by using simple sample with starting point $\vec{p}^{(1)}$. We iterate this procedure to $\lambda=\infty$. Our first test is identical to the system in Ref. [7] where multiple topplings [14] and grain redistribution [15] are not considered. The resultant fixed point $\vec{p}^{(\infty)}=(0,0.294,0.435,0.229,0.042)$ is very close to the one acquired through exact enumeration $\vec{p}^{(\infty)}=(0,0.295,0.435,0.229,0.041)$ [7]. Our second test [16] is the same as the system in Ref. [11] where grain redistribution is considered but multiple toppling is not considered. The resultant fixed point $\vec{p}^{(\infty)}=(0.090,0.345,0.379,0.161,0.025)$ is very close to the one acquired through exact enumeration $\vec{p}^{(\infty)}=(0.091,0.345,0.379,0.161,0.024)$ [7]. The above two cases provide evidence that our simple sampling RG scheme is valid and useful.

V. RESULTS AND DISCUSSIONS

This RG scheme has been polished by incorporating the critical exponent of area [6]. Suppose that an avalanche observed on the λth stage has $A^{(\lambda)} \geq 1$. The probability of this avalanche observed on the $(\lambda+1)$th stage corresponding to $A^{(\lambda+1)}=0$ is

$$G = \int_{b\Delta^{(\lambda)}}^{b\Delta^{(\lambda+1)}} \mathcal{P}(A^{(0)})dA^{(0)} \bigg/ \int_{b\Delta^{(\lambda)}}^{\infty} \mathcal{P}(A^{(0)})dA^{(0)} = 1-2^{2(1-\tau)}. \qquad (20)$$

Alternatively, G also corresponds the probability of a toppled λ square not being able to induce the topplings of its NN, i.e.,

$$G = \sum_{r=0}^{4} p_r^{(\lambda)}[1-n_4^{(\lambda)}]^r. \qquad (21)$$

At $\lambda=\infty$, the value of G is produced by putting the obtained $\vec{p}^{(\infty)}$ and $\vec{n}^{(\infty)}$ in Eq. (21). Subsequently, the value of τ will be obtained through Eq. (20).

The $\vec{n}^{(\infty)}$ and τ obtained from the first test of Sec. IV, where multiple topplings and grain redistribution are not considered, are listed in result (I) of Table I. The values of $\vec{n}^{(\infty)}$ and τ of previous RG [7] are also listed in Table I. In order to distinguish the effects of grain redistribution and multiple topplings, we simulated cases (II), (III), (IV) for considering grain redistribution only, multiple topplings only, and both grain redistribution and multiple topplings, respectively. We obtained $\vec{p}^{(\infty)}=(0.102,0.352,0.372,0.151,0.023)$, $\vec{n}^{(\infty)}=(0.013,0.100,0.313,0.574)$, and $\tau=1.291$ for case (II), $\vec{p}^{(\infty)}=(0,0.209,0.425,0.294,0.072)$, $\vec{n}^{(\infty)}=(0.032,0.164,0.355,0.449)$, and $\tau=1.258$ for case (III), and $\vec{p}^{(\infty)}=(0.024,0.219,0.415,0.277,0.065)$, $\vec{n}^{(\infty)}=(0.030,0.158,0.350,0.462)$, and $\tau=1.268$ for case (IV). All results are listed in Table I.

The corresponding exact solution of height probabilities [3], the predictions of exponent $\tau=1.25$ based on scaling argument [17], and $\tau=1.20$ [18] based on the simulations from the finite-size hypothesis are all listed in Table I. For height probabilities, result (III) is the closest to the exact values among results (I), (II), (III), and (IV). Furthermore, results (I) and (III) are better than results (II) and (IV), respectively. This implies that the appearance of the grain redistribution worsens the values of $\vec{n}^{(\infty)}$. On the other hand, results (III) and (IV) are better than results (I) and (II), respectively. This implies that the appearance of multiple topplings improves the values of $\vec{n}^{(\infty)}$. The obtained τ for these four cases are not far from $\tau=1.25$ [17] or $\tau=1.2$ [18].

In Ref. [19], the number of annihilated grains through a toppling is defined as the dissipation rate. Thus, in our RG scheme, $p_0^{(\lambda)}$ can be considered as the dissipation rate on the λth stage. Since Vespignani and Zapperi [19] stated that one of the requirements for a sandpile system exhibiting criticality is the dissipation rate being zero, it suggests that $p_0^{(\infty)}=0$ should hold. The above provides the reason why the results of (I) and (III) with $p_0^{(\infty)}=0$ are better than those of (II) and (IV) with $p_0^{(\infty)} \neq 0$, respectively.

We applied this scheme to a triangular lattice where each λ triangle has an area $\Delta^{(\lambda)}=3^{\lambda}$ [8] and six NNs. Then, a $(\lambda+1)$ triangle containing three λ triangles is expressed by RG

TABLE I. The height probabilities and τ for square (SQ) and triangular (TR) lattices.

SQ	$n_1^{(\infty)}$	$n_2^{(\infty)}$	$n_3^{(\infty)}$	$n_4^{(\infty)}$	—	—	τ
Exact solution [3]	0.074	0.174	0.306	0.446	—	—	—
Prediction [17]	—	—	—	—	—	—	1.25
Prediction [18]	—	—	—	—	—	—	1.20
Previous RG [7]	0.021	0.134	0.349	0.496	—	—	1.248
Our result (I)	0.021	0.134	0.349	0.496	—	—	1.248
Our result (II)	0.013	0.100	0.313	0.574	—	—	1.291
Our result (III)	0.032	0.164	0.355	0.449	—	—	1.258
Our result (IV)	0.030	0.158	0.350	0.462	—	—	1.268
TR	$n_1^{(\infty)}$	$n_2^{(\infty)}$	$n_3^{(\infty)}$	$n_4^{(\infty)}$	$n_5^{(\infty)}$	$n_6^{(\infty)}$	τ
Simulation	0.058	0.094	0.139	0.188	0.240	0.281	—
Previous RG [8]	0.036	0.135	0.198	0.210	0.211	0.211	1.367
Our result (I)	0.036	0.135	0.198	0.210	0.210	0.211	1.367

parameters $\vec{n}^{(\lambda)}=(n_1^{(\lambda)},n_2^{(\lambda)},n_3^{(\lambda)},n_4^{(\lambda)},n_5^{(\lambda)},n_6^{(\lambda)})$ and $\vec{p}^{(\lambda)}=(p_0^{(\lambda)},p_1^{(\lambda)},p_2^{(\lambda)},p_3^{(\lambda)},p_4^{(\lambda)},p_5^{(\lambda)},p_6^{(\lambda)})$. We adjust the equations from the version of a square lattice to those of a triangular lattice, e.g., Eqs. (10), (20), and (21) are changed to $n_j^{(\lambda)}=\sum_{j^+=6,-1}^{7-j}\theta_{j^+}/\sum_{j^+=1}^{6}j^+\theta_{j^+}$, $G=1-3^{(1-\tau)}$, and $G=\sum_{r=0}^{6}p_r^{(\lambda)}[1-n_6^{(\lambda)}]^r$, respectively.

For triangular cases, taking $\vec{p}^{(0)}=(0,0,0,0,0,0,1)$ and $t_c=10\,000$ for simple sampling, we consider cases (I), (II), (III), and (IV) as square cases. This obtains $\vec{p}^{(\infty)}=(0,0.0000334,0.00244,0.0571,0.296,0.471,0.173)$, (0, 0.0000334, 0.00244, 0.0571, 0.296, 0.471, 0.173), (0, 0.0000177, 0.00242, 0.0575, 0.296, 0.470, 0.174), and (0, 0.0000177, 0.00242, 0.0575, 0.296, 0.470, 0.174) for cases (I), (II), (III), and (IV), respectively. Case (I) is identical to the system of the previous study [8]. We list the $\vec{n}^{(\infty)}$ and τ for result (I) and the previous RG [8] in Table I. The consistency between these two results shows the reliability of our simple sampling procedure. Furthermore, the $\vec{n}^{(\infty)}$ and τ of results (II), (III), and (IV) are almost the same as those of result (I). It implies that both grain redistribution and multiple topplings have nearly no impact on the RG calculations. It is different from the square cases where both grain redistribution and multiple topplings affect the RG results. Since a $(\lambda+1)$ triangle contains only three λ triangles, but a $(\lambda+1)$ square contains four λ squares, we expect that multiple topplings and grain redistribution will play more important roles when a RG cell contains more subunits.

In summary, a numerical RG scheme for calculating the height probabilities and the area exponent of BTW sandpile is provided. Through the developed simple sampling procedure, it is achievable to incorporate multiple topplings and grain redistribution into RG equations. Compared with the previous RG, our RG improves the values of the fixed point of height probabilities for a square lattice. The use of our sampling method is well verified and has the potential for further RG investigations.

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[1] H. J. Jensen, *Self-Organized Criticality* (Cambridge University Press, New York, 1998).
[2] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990); S. N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).
[3] V. B. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
[4] R. J. Creswick, H. A. Farach, and C. P. Poole, Jr., *Introduction to Renormalization Group Methods in Physics* (Wiley, New York, 1992).
[5] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[6] L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994).
[7] E. V. Ivashkevich, Phys. Rev. Lett. **76**, 3368 (1996).
[8] Vl. V. Papoyan and A. M. Povolotsky, Physica A **246**, 241 (1997).
[9] J. Hasty and K. Wiesenfeld, Phys. Rev. Lett. **81**, 1722 (1998).
[10] Y. Moreno, J. B. Gomez, and A. F. Pacheco, Phys. Rev. E **60**, 7565 (1999).
[11] E. V. Ivashkevich, A. M. Povolotsky, A. Vespignani, and S. Zapperi, Phys. Rev. E **60**, 1239 (1999).
[12] We use the notation $n_4^{(\lambda)}$ to express $\rho^{(\lambda)}$ of Refs. [3,11].
[13] C.-Y. Lin and C.-K. Hu, Phys. Rev. E **66**, 021307 (2002).
[14] When one specified λ square topples once, we freeze this λ square. That is to prevent this λ square from the second toppling no matter what the height of that λ square is.
[15] In our RG scheme, the grain redistribution inside a RG cell of the λth stage induces a zero-directional toppling of the $(\lambda+1)$th stage. Not considering the grain redistribution is

achieved by the followings: (i) For the λth stage, keep $p_0^{(\lambda)}=0$. (ii) For the $(\lambda+1)$th stage, drop all active relaxations with $r'=0$ from $\{\Phi_a(H_a^*,I_a)\}$. This will force $p_0^{(\lambda+1)}=0$.

[16] In order to access the system in Refs. [3,11], we freeze all λ squares i with $h_i \leqslant 3$, i.e., such a λ square cannot topple forever even if the height later evolves to $h_i \geqslant 4$. This induces that Eqs. (8), (9), and (10) are not valid. However, Eq. (10) for $j=4$ still works because we consider it from the viewpoint of the balance between input flow and output flow.

[17] V. B. Priezzhev, D. V. Ktitarev, and E. V. Ivashkevich, Phys. Rev. Lett. **76**, 2093 (1996).

[18] C. Tebaldi, M. De Menech, and A. L. Stella, Phys. Rev. Lett. **83**, 3952 (1999).

[19] A. Vespignani and S. Zapperi, Phys. Rev. Lett. **78**, 4793 (1997); Phys. Rev. E **57**, 6345 (1998), and references therein.